

## Wolstenholme's Congruence for Lucas Sequences

The recent paper [1] in this Monthly showed that  $\sum_{k=1}^{p-1} \frac{1}{F_k^2} \equiv 0 \pmod{F_p}$  for any odd prime  $p$ , a variation of a well-known result of Wolstenholme. Below we show a similar result involving both Fibonacci numbers and Lucas numbers. Recall that the Lucas sequence  $L_n$  has the same recursive relationship as the Fibonacci sequence, but with different starting values, namely  $L_1 = 1, L_2 = 3$ . Here  $a/b \equiv 0 \pmod{n}$  means that  $\gcd(b, n) = 1$  and  $a \equiv 0 \pmod{n}$ .

**Theorem.** *If  $p$  is an odd prime, then  $\sum_{k=1}^{p-1} \frac{1}{L_k^2} \equiv 0 \pmod{F_p}$ .*

*Proof.* We rewrite the sum as follows:

$$\sum_{k=1}^{p-1} \frac{1}{L_k^2} = \sum_{k=1}^{\frac{p-1}{2}} \left( \frac{1}{L_k^2} + \frac{1}{L_{p-k}^2} \right) = \sum_{k=1}^{\frac{p-1}{2}} \frac{L_k^2 + L_{p-k}^2}{L_k^2 L_{p-k}^2}.$$

It suffices to show that each term of this sum is congruent to 0  $\pmod{F_p}$ . Let  $1 \leq k \leq \frac{p-1}{2}$ . We first show that each denominator is coprime to  $F_p$ . Since  $p$  is an odd prime, we have that  $\gcd(2k, p) = 1$ . By strong divisibility of the Fibonacci sequence, we get  $\gcd(F_{2k}, F_p) = F_1 = 1$ . Applying the identity  $F_{2k} = F_k L_k$  then yields  $\gcd(L_k, F_p) = 1$ ; hence  $\gcd(L_k^2 L_{p-k}^2, F_p) = 1$  as required.

We now show that  $L_k^2 + L_{p-k}^2 \equiv 0 \pmod{F_p}$ . Applying the identity  $L_n = F_{n+1} + F_{n-1}$  gives

$$\begin{aligned} L_k^2 + L_{p-k}^2 &= (F_{k+1} + F_{k-1})^2 + (F_{p-k-1} + F_{p-k+1})^2 \\ &= (F_{k+1}^2 + F_{p-(k+1)}^2) + (F_{k-1}^2 + F_{p-(k-1)}^2) \\ &\quad + 2(F_{k+1}F_{k-1} + F_{p-k-1}F_{p-k+1}). \end{aligned} \quad (1)$$

Applying Cassini's identity,  $F_{n-1}F_{n+1} = (-1)^n + F_n^2$ , to the final term gives

$$F_{k+1}F_{k-1} + F_{p-k-1}F_{p-k+1} = (F_k^2 + F_{p-k}^2) + ((-1)^k + (-1)^{p-k}). \quad (2)$$

Since  $p$  is odd, it follows from Catalan's identity that  $F_n^2 + F_{p-n}^2 = F_p F_{p-2n}$  for all  $n \in \mathbb{N}$ . This implies that the first two terms of (1), as well as the first term of the right-hand side of (2), are congruent to 0  $\pmod{F_p}$ . Therefore  $L_k^2 + L_{p-k}^2 \equiv 2((-1)^k + (-1)^{p-k}) \equiv 2(-1)^k(1 + (-1)^p) = 0 \pmod{F_p}$ . ■

### REFERENCES

- [1] B. He, Y. L. Mao & A. Togbé. (2023). A Fibonacci Version of Wolstenholme's Harmonic Series Congruence, Amer. Math. Monthly, 130:1, 83-85, DOI: 10.1080/00029890.2022.2128165

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